NOTES ON THE ARTICLE "SOME METHODS OF SOLUTION OF NONLINEAR PROBLEMS OF THE MECHANICS OF DEFORMABLE SOLIDS"

(ZAMECHANIIA K STAT'E "NEKOTORYE METODY RESHENIIA NELINEINYKH ZADACH MEKHANIKI DEFORMIRUEMOGO TELA")

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In the paper [1] certain methods of solution were considered (*) for nonlinear equations of the form (1.3)*

$$Ax = 0 \qquad (Ax = \mathrm{grad}\, f(x)) \tag{1}$$

where $f(x)$ is some functional given in a real Hilbert space H. For the solution of Equation (1), in particular, use was made of the process (2.2)*

$$x^{(\nu+1)} = x^{(\nu)} - \alpha B^{-1} A x^{(\nu)} \qquad (\alpha = \mathrm{const} > 0) \tag{2}$$

where B is a certain positive-definite operator satisfying condition (2.4)*.

Here we will consider the rapidity of convergence of process (2), and also that class of functions for which this process converges when A and B are differential operators. We will begin with the second question.

We will assume that in the appropriate space the differential operator A has a derivative $A'(x)$ whose form will be described in the following. Let the coefficients of the differential operator $A'(x)$ be bounded functions and the order of its highest derivative be $2n$.

As the Hilbert space H we will take the space $W_2^{(n)}$ of functions whose nth derivatives are L_2 functions.

Let us consider the quadratic functional $F(x, h) = (A'(x) h, h)$. If the contour integrals for the integration by parts vanish, $F(x,h)$ contains derivatives of function h of order no higher than the nth. Hence, under the conditions adopted, $F(x,h)$ will be a bounded functional in $W_2^{(n)}$. If, however, it is also continuous with respect to x, the operator $A'(x)$ will be symmetric [2]. From the boundedness of the functional $F(x,h)$, it follows that $A'(x)$ is also bounded and is therefore the derivative of a Fréchet operator A. Hence, the latter is also continuous in $W_2^{(n)}$ [3]. As can be seen from (1.2)*, the function $f(x)$ is also continuous in $W_2^{(n)}$.

Let B be a differential operator of the same order as $A'(x)$. We will

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*) For brevity, the asterisk will denote the number given to the corresponding formula in [1].

choose its coefficients so that, besides condition (2.4)*, they also satisfy the following conditions:

$$m \| h \| \leqslant (Bh, h), \qquad |(Bh_1, h_2)| \leqslant M \| h_1 \| \| h_2 \| \qquad (h, h_1, h_2 \in H_1 \subset W_2^{(n)}) \qquad (3)$$

The norm in (3) is taken in $W_2^{(n)}$ Then, from (2.6)* we obtain

$$\| x^{(\nu)} - x^{(\nu+1)} \| \leqslant \frac{1}{m}(Bh^{(\nu)}, h^{(\nu)}) \leqslant \frac{1}{m\mu} [f(x^{(\nu)}) - f(x^{(\nu+1)})] \qquad \left(\mu = \frac{1}{\alpha} - \frac{1}{2} K > 0\right)$$

whence it follows that

$$\| x^{(\nu+p)} - x^{(\nu)} \|_{W_2^{(n)}} \leqslant \frac{1}{m\mu} [f(x^{(\nu)}) - f(x^{(\nu+p)})]$$

Since $\{f(x^{(\nu)})\}$ is a convergent sequence, $\{x^{(\nu)}\}$ converges in the mean, i.e. to a limit element $x' \in W_2^{(n)}$

Similar to [4], a function x^* that satisfies the condition

$$(Ax^*, h) = 0 \qquad (4)$$

for arbitrary $h \in H_1$ will be called a generalized solution of equation (1). The left-hand side of (4) is regarded as the first variation of the functional $f(x)$.

Here also we will assume that the contour integral vanishes. Then, (2) can be written in the form

$$(Bh^{(\nu)}, h) = -\alpha(Ax^{(\nu)}, h) \qquad (h \in H_1) \qquad (5)$$

Since the sequence $\{h^{(\nu)}\}$ tends to zero [1] in $W_2^{(n)}$, it follows from (3) that the left-hand side of Equation (5) tends to zero as $\nu \to \infty$. Then, keeping the continuity of operator A in mind, we find from (5) that $x^{(\nu)} \to x'$ implies $(Ax^{(\nu)}, h) \to (Ax', h) = 0$, i.e. $x' = x^*$ will be the generalized solution of Equation (1).

Now let $f(x)$ be given in space H. We will prove that if the operator B in H satisfies condition (3) and moreover the following condition is fulfilled

$$0 < \gamma \| h \|^2 \leqslant W(x, h) \leqslant KM \| h \|^2 \qquad (x, h \in H) \qquad (6)$$

where the constants γ, $K > 0$ ($\gamma < M$), then process (2) converges with the rapidity of a geometric progression with common ratio $|q| < 1$. The optimum value of the coefficient α for which the best convergence is obtained, is a certain value $\alpha > 1/K$.

From (2.5)* and (2), remembering that $(B^{-1}x, x) \geqslant M^{-1} \| x \|^2$, we obtain

$$f(x^{(\nu)}) - f(x^{(\nu+1)}) \geqslant \alpha (1 - {}^1/_2\, \alpha K)(B^{-1} Ax^{(\nu)}, Ax^{(\nu)}) \geqslant \alpha M^{-1} (1 - {}^1/_2\, \alpha K) \| Ax^{(\nu)} \|^2 \qquad (7)$$

On the other hand

$$f(x^*) - f(x) = -(Ax, x - x^*) + {}^1/_2 W(x, x - x^*)$$
$$f(x) - f(x^*) = {}^1/_2 W(x^*, x - x^*)$$

Hence we obtain

$${}^1/_2 W(x^*, x - x^*) = (Ax, x - x^*) - {}^1/_2 W(x, x - x^*) \leqslant (Ax, x - x^*)$$

and therefore

$$f(x) - f(x^*) \leqslant (Ax, x - x^*) \qquad (8)$$

We will represent condition (6) as

$$\gamma \| h \|^2 \leqslant (A(x + h) - Ax, h) \leqslant KM \| h \|^2$$

Then $(Ax, h) = (Ax - Ax^*, h) \geqslant \gamma (h, h)$ and from the inequality $(Ax, h)^2 \leqslant (Ax, Ax)(h, h)$ it follows that $|(Ax, h)| \leqslant \gamma^{-1} \| Ax \|^2$. Substituting the latter into (8), we obtain

$$f(x) - f(x^*) \leqslant \gamma^{-1} \| Ax \|^2 \qquad (9)$$

We will introduce [5] the notation $\varphi(x) = f(x) - f(x^*)$. Then, from (7) and (9) it follows that

$$\varphi(x^{(\nu)}) - \varphi(x^{(\nu+1)}) \geqslant \gamma M^{-1} \alpha (1 - {}^1/_2\, \alpha K)\, \varphi(x^{(\nu)})$$

whence it follows that $q\varphi(x^{(\nu)}) \geqslant \varphi(x^{(\nu+1)})$, where $q = 1 - \gamma M^{-1}\alpha(1 - {}^1/_2\,\alpha K)$. If $0 < \varepsilon_1 \leqslant \alpha \leqslant \varepsilon_2 < 2/K$, then $q < 1$, and we obtain

$$f(x^{(n)}) - f(x^*) \leqslant q^n [f(x^{(0)}) - f(x^*)] \qquad (10)$$

i.e. $\{f(x^{(\nu)})$ tends to $f(x^*)$ with the rapidity of a geometric progression. The constant q attains its smallest value, $q = 1 - \gamma/2MK$, when $\alpha = 1/K$. The constant K is usually set too large, and therefore it is necessary to take α somewhat larger than $1/K$.

From (2.6)* it follows that

$$(Bh^{(\nu)}, h^{(\nu)}) \leqslant \left(\frac{1}{\alpha} - \frac{1}{2}K\right)^{-1} [f(x^{(\nu)}) - f(x^{(\nu+1)})] \leqslant \left(\frac{1}{\alpha} - \frac{1}{2}K\right)^{-1} \varphi(x^{(\nu)})$$

or

$$\|h^{(\nu)}\| \leqslant \frac{1}{m}\left(\frac{1}{\alpha} - \frac{1}{2}K\right)^{-1} \varphi(x^{(\nu)}) = r\varphi(x^{(\nu)})$$

Then

$$\|x^{(n+p)} - x^{(n)}\| \leqslant \sum_{\nu=n}^{n+p-1} \|h^{(\nu)}\| \leqslant r\varphi(x^{(0)}) \sum_{\nu=n}^{n+p-1} q^\nu$$

which, since $q < 1$, leads to

$$\|x^{(n)} - x^*\| \leqslant \frac{1}{m}\left(\frac{1}{\alpha} - \frac{1}{2}K\right)^{-1} \frac{q^n}{1-q}\varphi(x^{(0)})$$

Thus, the convergence of $\{x^{(\nu)}\}$ is just as fast as that of $\{f(x^{(\nu)})\}$.

As we noted in [1], process (2) generalizes the modified Newton's method. In general, the latter method converges only when the initial approximation $x^{(0)}$ is sufficiently close to x^* [3]. On the other hand, in the present case of potential operators, process (2) converges to the solution of Equation (1) independently of the initial approximation.

In [1] there is presented an investigation of the convergence of the Galerkin method, which was then used to prove the convergence of the method of partial approximations. For a proof of the first method certain rather rigid assumptions were made, which proved to be superfluous. However, the convergence of the Galerkin method in the case of equations with a potential operator will be a consequence of the convergence of the Ritz method treated in [6]. Hence, this question has not been dealt with here.

BIBLIOGRAPHY

1. Simeonov, S.V., Nekotorye metody resheniia nelineinykh zadach mekhaniki deformiruemogo tela (Certain methods of solution of nonlinear problems of the mechanics of deformable solids). *PMM* Vol.28, № 3, 1964.
2. Vainberg, M.M., Variatsionnye metody issledovaniia nelineinykh operatorov (Variational Methods of Studying Nonlinear Operators). Gostekhizdat,1956.
3. Kantorovich, L.V. and Akilov, G.P., Funktsional'nyi analiz v normirovannykh prostranstvakh (Functional Analysis in Normed Spaces). Fizmatgiz, 1959.
4. Vorovich, I.I. and Krasovskii, Iu.P., O metode uprugikh reshenii (On the method of elastic solutions). Dokl.Akad.Nauk SSSR, Vol.126, № 4,1959.
5. Poliak, B.T., Gradientnye metody minimizatsii funktsionalov (Gradient methods of minimizing functionals). Zh.vychisl.matem. i matem.fiziki, Vol.3, № 4, 1963.
6. Mikhlin, S.G., O metode Rittsa v nelineinykh zadachakh (On the Ritts method in nonlinear problems). Dokl.Akad.Nauk SSSR, Vol.142, № 4, 1962.

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